The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2 ***Solutions Version 2.0***

Date: November 14, 2024 Course: EE 313 Evans

Name: ____
Last, First

- **Exam duration**. The exam is scheduled to last 75 minutes.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **No AI tools allowed**. As mentioned on the course syllabus, you may not use GPT or other AI tools during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except the proctor for the test, and that you did not provide help, directly or indirectly, to another student taking this exam.

Problem	*Point Value*	*Your score*	*Topic*
1	27		System Properties
2	29		FIR Filter Analysis
3	24		System Identification
4	20		Filter Design
Total	100		

SPFirst Sec. 5-4, 5-5, 8-2 & 8-4.2	*Handout H on BIBO Stability*
Lecture Slides 8-3 to 8-8 and 11-12 to 11-13	*HW 5.2*
Midterm #2 Problems: 2.5 in F18, 2.1 in F21 & 2.1 in F23	

Problem 2.1. *System Properties*. 27 points.

Each discrete-time system has input $x[n]$ and output $y[n]$, and $x[n]$ and $y[n]$ might be complex-valued.

Determine if each system is linear or nonlinear, time-invariant or time-varying, and bounded-input bounded-output (BIBO) stable or unstable.

You must either prove that the system property holds in the case of linearity, time-invariance, or stability, or provide a counter-example that the property does not hold. Providing an answer without any justification will earn 0 points.

Part	*System Name*	*System Formula*	*Linear?*	*Time-Invariant?*	*BIBO Stable?*
(a)	Add a DC Offset	$y[n] = x[n] + 1$ for $-\infty < n < \infty$	**No**	**Yes**	**Yes**
(b)	Reciprocal	$y[n] = \dfrac{1}{x[n]}$ for $-\infty < n < \infty$	**No**	**Yes**	**No**
(c)	Scaling of the time axis	$y[n] = x[2\,n]$ for $-\infty < n < \infty$	**Yes**	**No**	**Yes**

***Linearity*. We'll first apply the all-zero input test. If the output is not zero for all time, then the system is not linear. Otherwise, we'll have to apply the definitions for homogeneity and additivity. All-zero input test is a special case of homogeneity $a\, x[n] \rightarrow a\, y[n]$ when the constant $a = 0$.**

***BIBO Stability*. Bounded input $|\, x[n]\,| \le B < \infty$ would give bounded output $|\, y[n]\,| \le C < \infty$.**

(a) Add a DC offset to the input signal: $y[n] = x[n] + 1$ for $-\infty < n < \infty$. *9 points*.

***Linearity*: Does not pass all-zero input test. When $x[n] = 0$ for $-\infty < n < \infty$, $y[n] = 1$. NO.**

***Time-Invariance*: Pointwise operation; current output value $y[n]$ depends only on current input $x[n]$ and not on any other input/output values. All pointwise operations are time-invariant. YES.**

***BIBO Stability*. $|\, y[n]\,| = |\, x[n] + 1\,| \le |\, x[n]| + 1 = B + 1 < \infty$. YES.**

(b) Reciprocal: $y[n] = \frac{1}{x[n]}$ for $-\infty < n < \infty$

***Linearity*: Does not pass the all-zero input test. When $x[n] = 0$ for $-\infty < n < \infty$, $y[n] = \frac{1}{0}$. If we take the limit as $x[n] \rightarrow 0$, then $y[n] \rightarrow \infty$. NO.**

***Time-Invariance*: Pointwise operation; current output value $y[n]$ depends only on current input $x[n]$ and not on any other input/output values. All pointwise operations are time-invariant. YES.**

***BIBO Stability*. When $x[n] = 0$ for $-\infty < n < \infty$, $y[n] \rightarrow \infty$ in the limit. No bounded. NO.**

(c) Scaling of the time axis: $y[n] = x[2\,n]$ for $-\infty < n < \infty$ *9 points*.

***Linearity*: Passes the all-zero input test.**

- ***Homogeneity:* $y_{scaled}[n] = (a\, x[n])_{n \rightarrow 2n} = a\, x[2n] = a\, y[n]$. YES.**
- ***Additivity*: $y_{additive}(t) = (x_1[n] + x_2[n])_{n \rightarrow 2n} = x_1[2n] + x_2[2n] = y_1[n] + y_2[n]$. YES.**

***Time-Invariance*: $y[n] = x[2n]$ selects every even-indexed value of $x[n]$: $\{\, \ldots, x[-2], x[0], x[2], \ldots.\}$.**
Input $x[n-1]$. Output $y_{shifted}[n]$ will be $\{\, \ldots, x[-3], x[-1], x[1], \ldots.\}$. This is not $y[n-1]$. NO.

***BIBO Stability*. $|\, y[n]\,| = |\, x[2\,n]\,| \le B < \infty$. YES.**

System in (c) is called downsampling by 2 because it keeps every other value of the input signal. This operation is used to reduce the sampling rate of $x[n]$ by a factor of 2. It's a building block in convolutional neural networks, communication receivers and many other systems.

SPFirst Sec. 6-1 to 6-6 & 8-2 to 8-6 | Handout H on BIBO Stability
Lecture Slides 8-3 to 8-8 and 11-12 to 11-13 | HW 4.2, 6.1, 7.1 & 7.2
Fall 2018 Midterm Problem 2.3 | Tuneup #7 | Mini-Project #2

Problem 2.2 *FIR Filter Analysis*. 29 points.

Consider a causal linear time-invariant (LTI) discrete-time finite impulse response (FIR) filter with input $x[n]$ and output $y[n]$ observed for $n \geq 0$ governed by

This is the nulling filter from mini-project #2

$$y[n] = x[n] + b\, x[n-1] + x[n-2] \quad \text{for } n \geq 0$$

where $b = -2\cos(\hat{\omega}_0)$ and $\hat{\omega}_0$ is a constant that is discrete-time frequency in units of rad/sample.

(a) What are the initial condition(s) and their value(s)? Why? *5 points*.

We can see the initial conditions by starting to compute the first few values of $y[n]$.

SPFirst Sec. 8-2.1 on page 199

$$y[0] = x[0] + b\, x[-1] + x[-2]$$
$$y[1] = x[1] + b\, x[0] + x[-1]$$
$$y[2] = x[2] + b\, x[1] + x[0]$$

The initial conditions need to be zero to satisfy the necessary (but not sufficient) conditions for linearity and time-invariance properties to hold.

$$x[-1] = x[-2] = 0$$

(b) Derive a formula for the transfer function in the z-domain including the region of convergence. *6 points*.

Take the z-transform of both sides of difference equation with initial conditions being zero:

$$Y(z) = X(z) + b\, z^{-1} X(z) + z^{-2} X(z)$$

Divide both sides by $X(z)$ to obtain the system transfer function $H(z)$ in the z-domain:

$$\frac{Y(z)}{X(z)} = 1 + b\, z^{-1} + z^{-2}$$

The region of convergence has to exclude divisions by zero: $z \neq 0$

(c) Derive a formula for the discrete-time frequency response of the filter. Justify your approach. *6 points*.

Since the transfer function $H(z)$ includes the unit circle in the region of convergence, we can substitute $z = e^{j\hat{\omega}}$ to convert the transfer function into a frequency response:

$$H(e^{j\hat{\omega}}) = 1 + b\, e^{-j\hat{\omega}} + e^{-2j\hat{\omega}}$$

(d) Which best describes the filter's magnitude response and why? Lowpass, highpass, bandpass, bandstop, notch/nulling, or allpass. *6 points*.

<u>Answer #1</u>: **The system transfer function $H(z)$ in the z-domain has two zeros. Using the quadratic formula, the zeros are at locations**

$$\frac{-b \pm \sqrt{b^2 - 4}}{2}$$

After substituting $b = -2\cos(\hat{\omega}_0)$,

$$\frac{2\cos(\hat{\omega}_0) \pm \sqrt{4\cos^2(\hat{\omega}_0) - 4}}{2} = \frac{2\cos(\hat{\omega}_0) \pm 2\sqrt{\cos^2(\hat{\omega}_0) - 1}}{2} = \cos(\hat{\omega}_0) \pm \sqrt{-\sin^2(\hat{\omega}_0)}$$

which is $\cos(\hat{\omega}_0) \pm j\sin(\hat{\omega}_0)$. This is a nulling filter that removes frequencies at $\pm\hat{\omega}_0$.

<u>Answer #2</u>: **This is the nulling filter from mini-project #2 which removes frequencies at $\pm\hat{\omega}_0$.**

(e) Give all possible conditions on the constant b so that the FIR filter has constant group delay (i.e. linear phase). Compute the constant group delay. *6 points*.

***Answer #1*:** **The group delay is the negative of the derivative with respect to frequency of the phase of the discrete-time frequency response.**

Let's factor the frequency response into an amplitude and phase:

$$H\left(e^{j\hat{\omega}}\right) = 1 + b\, e^{-j\hat{\omega}} + e^{-2j\hat{\omega}} = e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + b + e^{-j\hat{\omega}}\right) = e^{-j\hat{\omega}}\,(b + \cos(\hat{\omega}))$$

$$H\left(e^{j\hat{\omega}}\right) = \underbrace{(b + 2\cos(\hat{\omega}))}_{amplitude\ term}\ \underbrace{e^{-j\hat{\omega}}}_{phase\ term}$$

The phase $\angle H\left(e^{j\hat{\omega}}\right) = -\hat{\omega}$ **and**

$$\text{Group Delay}(\hat{\omega}) = -\frac{d}{d\hat{\omega}}\angle H\left(e^{j\hat{\omega}}\right) = 1$$

This is true for all values of b.

The amplitude term can be negative, zero, and positive. We can take the absolute value of the amplitude term to find the magnitude response. For the range of frequency values for which the amplitude function is negative, the magnitude is computed by multiplying the amplitude function by -1. This corresponds to a phase shift of π because $-1 = e^{j\pi}$. This does not affect the calculation of the group delay since the derivative of a constant is zero.

***Answer #2*:** **The impulse response is $h[n] = \delta[n] + b\,\delta[n-1] + \delta[n-2]$ which is even symmetric about its midpoint at index $n = 1$. The constant group delay is 1 sample. This is true for all values of b.**

Problem 2.3 *System Identification.* 24 points.

You are given several causal discrete-time linear time-invariant (LTI) systems each with unknown impulse response $h[n]$ but you are able to observe the input signal $x[n]$ and output signal $y[n]$ for $-\infty < n < \infty$.

For reference, the unit step function $u[n]$ is defined as

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$y[n]$	$Y(z)$	Region of Convergence
$\delta[n]$	1	all z
$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$(n+1)\, u[n]$	$\left(\frac{1}{1-z^{-1}}\right)^2$	$\|z\| > 1$
$a^n\, u[n]$	$\frac{1}{1-a\,z^{-1}}$	$\|z\| > \|a\|$

(a) When the input is $x[n] = \delta[n] - \delta[n-1]$, the output is

$$y[n] = \delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3]$$

i. Find the *finite* impulse response $h[n]$. *6 points.*

Answer #1: **We can solve for $H(z)$ and then take the inverse z-transform. In the z-domain, $Y(z) = H(z)\, X(z)$; $i.e.$**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1} - z^{-2} + z^{-3}}{1 - z^{-1}}$$

We can use the Matlab command `roots( [1 -1 -1 1] )` **to find that the numerator polynomial has roots $\{-1, 1, 1\}$:**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - z^{-1})(1 - z^{-1})(1 + z^{-1})}{1 - z^{-1}} = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}$$

(Or we could use polynomial long division of $1 - z^{-1}$ into $1 - z^{-1} - z^{-2} + z^{-3}$.) Taking the inverse z-transform of $H(z)$ gives $h[n] = \delta[n] - \delta[n-2]$.

Answer #2: **The convolution of two causal signals gives a causal result. Since $x[n]$ and $y[n]$ are causal, $h[n]$ must be causal, i.e. $h[n] = 0$ for $n < 0$. Furthermore, when convolving two finite length signals, the result is the sum of the lengths minus 1. Since $x[n]$ has 2 samples and $y[n]$ has 4 samples, $h[n]$ has 3 samples:**

$$y[n] = h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2]$$

We can use a deconvolution approach:

$$y[0] = h[0]\, x[0] \text{ which means } h[0] = \frac{y[0]}{x[0]} = \frac{1}{1} = 1$$

$$y[1] = h[0]\, x[1] + h[1]\, x[0] \text{ which means } h[1] = \frac{y[1] - h[0]\, x[1]}{x[0]} = \frac{-1+1}{1} = 0$$

$$y[2] = h[0]\, x[2] + h[1]\, x[1] + h[2]\, x[2] \text{ which means } h[2] = -1$$

ii. Verify your answer by convolving $h[n]$ and $x[n]$. *6 points.*

Answer #1: **We can compute the convolution in the z-domain using**

$$Y(z) = H(z)\, X(z) = (1 - z^{-1})(1 - z^{-2}) = 1 - z^{-1} - z^{-2} + z^{-3}$$

Answer #2: **We can compute the convolution in the time domain: $y[n] = h[n] * x[n]$. Using Matlab,** `conv( [1 -1], [1 0 -1] )` **gives** `[1 -1 -1 1]`.

(b) When the input is $x[n] = u[n]$, the output is $y[n] = (n+1)\, u[n]$.

i. Find the *infinite* impulse response $h[n]$. *6 points*.

Answer #1: **We can solve for $H(z)$ and then take the inverse z-transform. In the z-domain, $Y(z) = H(z)\, X(z)$; i.e.**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\frac{1}{1-z^{-1}}\right)^2}{\frac{1}{1-z^{-1}}} = \frac{1}{1-z^{-1}}$$

Taking the inverse z-transform gives $h[n] = u[n]$.

Answer #2: **The convolution of two causal signals gives a causal result. Since $x[n]$ and $y[n]$ are causal, $h[n]$ must be causal, i.e. $h[n] = 0$ for $n < 0$. Since $x[n]$ and $y[n]$ are infinite in length, $h[n]$ could be finite or infinite in length:**

$$y[n] = h[0]\, x[n] + h[1]\, x[n-1] + h[2]\, x[n-2] + \cdots$$

We can use a deconvolution approach:

$$y[0] = h[0]\, x[0] \text{ which means } h[0] = \frac{y[0]}{x[0]} = \frac{1}{1} = 1$$

$$y[1] = h[0]\, x[1] + h[1]\, x[0] \text{ which means } h[1] = \frac{y[1] - h[0]\, x[1]}{x[0]} = \frac{2-1}{1} = 1$$

$$y[2] = h[0]\, x[2] + h[1]\, x[1] + h[2]\, x[2] \text{ which means } h[2] = 1$$

We could infer that $h[n] = u[n]$.

ii. Verify your answer by convolving $h[n]$ and $x[n]$. *6 points*.

Answer #1: **We can compute the convolution in the z-domain using**

$$Y(z) = H(z)\, X(z) = \left(\frac{1}{1-z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right) = \left(\frac{1}{1-z^{-1}}\right)^2$$

The inverse z-transform of $Y(z)$ is $y[n] = (n+1)\, u[n]$.

Answer #2: **We can compute the convolution in the discrete-time domain:**

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]\, x[n-m] = \sum_{m=-\infty}^{\infty} u[m]\, u[n-m]$$

Note that $u[m] = 0$ for $m < 0$. Hence, the lower limit can be replaced by 0.

Note that $u[n-m] = 0$ for $n - m < 0$ *or equivalent* $m < n$. Hence, the upper limit can be replaced by n when $n \geq 0$

$$y[n] = \sum_{m=-\infty}^{\infty} u[m]\, u[n-m] = \sum_{m=0}^{n} 1 = (n+1) \quad \text{if } n > 0$$

In other words, $y[n] = (n+1)\, u[n]$.

SPFirst Ch. 6 & 7	SPFirst Sec. 8-4, 8-5, 8-6, 8-9 & 8-10	Tuneup #7
Lecture Slides 10-9 to 10-11 and 11-5 to 11-11	HW 5.4, 6.1, 7.1, 7.2 & 7.3	
Midterm #2: 2.3 & 2.4 on F17; 2.3 on F18; 2.3 on F21; and 2.4 on F23		

Problem 2.4. *Filter Design*. 20 points.

Consider designing discrete-time linear time-invariant (LTI) infinite impulse response (IIR) filters.

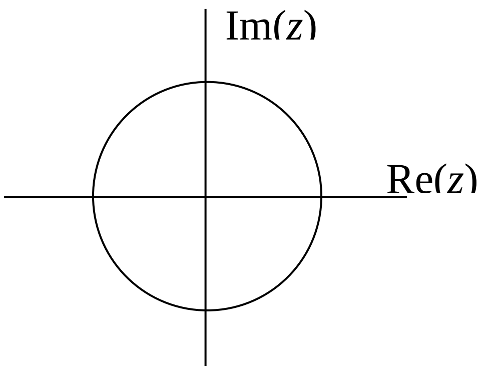
A second-order LTI IIR filter has zeros z_0 and z_1 and poles p_0 and p_1, and its transfer function in the z-domain (where C is a constant) is

$$H(z) = C\,\frac{(z - z_0)(z - z_1)}{(z - p_0)(z - p_1)}$$

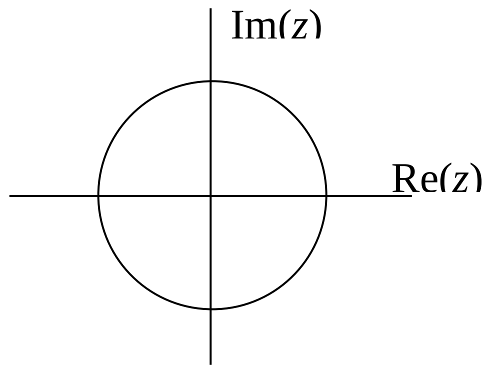
In this problem, **the poles and zeros will be complex-valued but <u>not real-valued</u>.** The imaginary part of the complex number cannot be zero, and the real part of the complex number can be anything.

Give numeric values for zeros z_0 and z_1 and poles p_0 and p_1 to give each magnitude response below, place the zeros and poles on the pole-zero diagram, and explain your reasoning. Please use O to indicate zero locations and X to indicate pole locations. **For each part, each zero and each pole must have a unique value. No two can have the same value.**

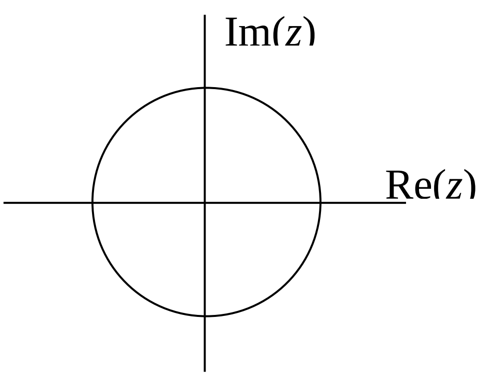
(a) Lowpass filter



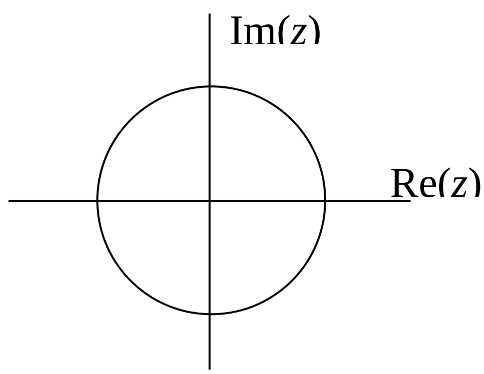
(b) Highpass filter



(c) Allpass filter



(d) Notch filter



Per [lecture slide 11-7](),

- **Angle of pole near unit circle indicates frequency at which peak occurs in magnitude response**
- **Angle of zero on or near unit circle indicates frequency at which valley occurs in mag. response.**

Although not explicitly requested, we will choose poles that are *conjugate symmetric* to give *real-valued* feedback coefficients, and zeros that are *conjugate symmetric* to give *real-valued* feedforward coefficients, as we've been doing throughout the semester.

Another reason for poles and zeros to be conjugate symmetric is the following. Consider an input signal that is a cosine at fixed frequency $\hat{\omega}_0$:

$$x[n] = \cos(\hat{\omega}_0\, n) = \frac{1}{2} e^{-j\,\hat{\omega}_0\, n} + \frac{1}{2} e^{j\,\hat{\omega}_0\, n}$$

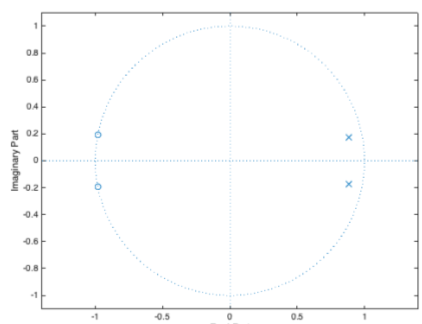
which has conjugate symmetric complex sinusoids at frequencies $-\hat{\omega}_0$ and $+\hat{\omega}_0$.

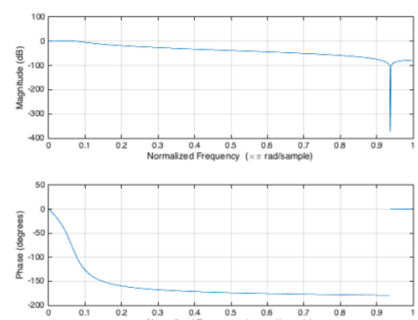
Lecture slide 11-10

```
%%% Lowpass filter example
zeroAngle = 15*pi/16;
z0 = exp(j*zeroAngle);
z1 = exp(-j*zeroAngle);
numer = [1 -(z0+z1) z0*z1];

r = 0.9;
poleAngle = pi/16;
p0 = r * exp(j*poleAngle);
p1 = r * exp(-j*poleAngle);
denom = [1 -(p0+p1) p0*p1];

%%% Normalize frequency response
%%% to 1 at center of passband
z = 1; zvec = [1 z^(-1) z^(-2)]';
C = (denom * zvec) / (numer * zvec);

figure; zplane(C*numer, denom);
figure; freqz(C*numer, denom);
```

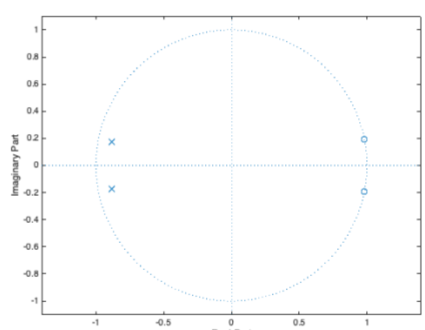
For lowpass and highpass filters, we'll place each pole at the center of each passband as per [lecture slide 11-10]() and *DSP First* [Three-Domain Connections demo]() in [lecture slide 11-11]().

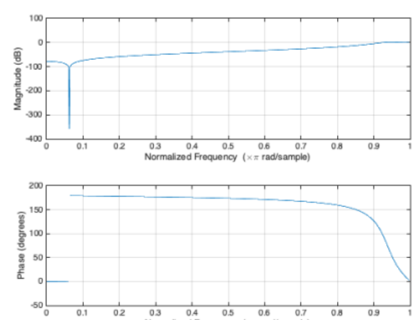
(a) Lowpass: zeros at
$z_0 = e^{j\theta_1}$ and
$z_1 = e^{-j\theta_1}$
with $\theta_1 = \frac{15}{16}\pi$ and
poles at
$p_0 = 0.9\, e^{j\theta_2}$ and
$p_1 = 0.9\, e^{-j\theta_2}$
with $\theta_2 = \frac{1}{16}\pi$





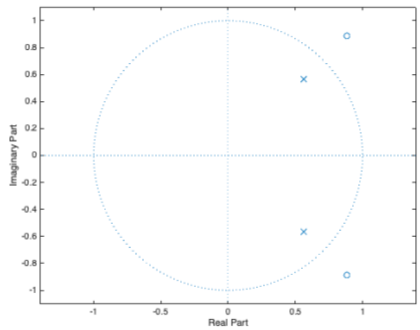
(b) Highpass: zeros at
$z_0 = e^{j\theta_2}$ and
$z_1 = e^{-j\theta_2}$
with $\theta_2 = \frac{1}{16}\pi$ and
poles at
$p_0 = 0.9\, e^{j\theta_1}$ and
$p_1 = 0.9\, e^{-j\theta_1}$
with $\theta_1 = \frac{15}{16}\pi$

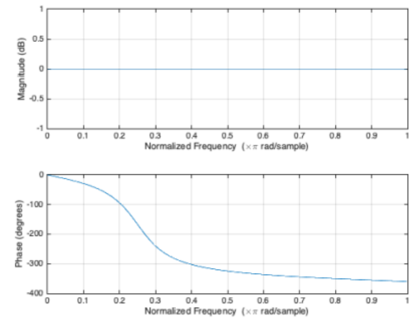




For allpass filters, we'll follow [all-pass filter handout] to place pole-zero pairs at the same angle and reciprocal magnitudes.

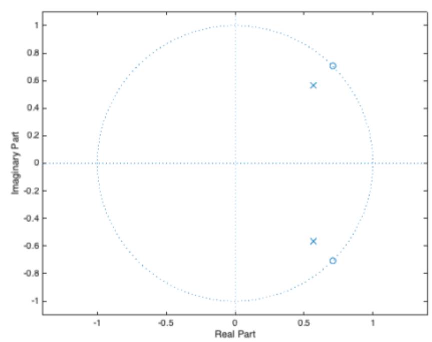
(c) Allpass: zeros at
$z_0 = 1.25\, e^{j\theta_4}$ and
$z_1 = 1.25\, e^{-j\theta_4}$
with $\theta_4 = \frac{\pi}{4}$
poles at
$p_0 = 0.8\, e^{j\theta_4}$ and
$p_1 = 0.8\, e^{-j\theta_4}$

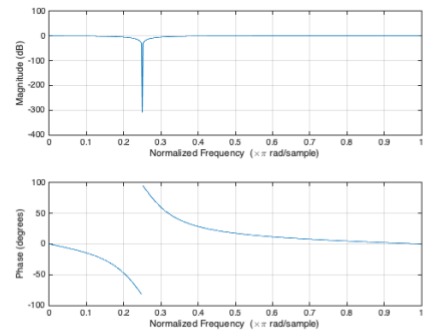




For notch filters, we'll follow the nulling filter analysis from mini-project #2 to place the zeros on the unit circle. We'll generalize the pole placements based on the first-order DC notch filter at the 0:04 mark in the *DSP First* [Three-Domain Connections demo] "IIR filter with one pole and one zero."

(d) Notch: zeros at
$z_0 = e^{j\theta_4}$ and
$z_1 = e^{-j\theta_4}$
with $\theta_4 = \frac{\pi}{4}$
poles at
$p_0 = 0.8\, e^{j\theta_4}$ and
$p_1 = 0.8\, e^{-j\theta_4}$





Selectivity	**Example Application(s)**	**Selectivity**	**Example Application**
Lowpass	Anti-aliasing filter before sampler in ADC; demodulation	Bandstop	
Highpass	Enhance edges/texture in images	Allpass	Phase correction
Bandpass	Modulation	Notch	Remove a specific unwanted frequency as in mini-project #2